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Generalised regression theorem and correlation functions of fluctuation operators in quantum optics

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Abstract. We consider a dynamic system interacting with a special dissipative system composed of many harmonic oscillators. Using the Feynman disentangling theorem and a bath approximation we derive differential equations for correlation functions of dynamic operators (CFD) concerning the maximum time argument t which in general occurs in more than one operator. They must be solved with appropriate initial conditions at time t', the second in height time argument in the CFD. This procedure allows the successive calculation of CFD with an arbitrary time arrangement (e.g. CFD of three operators, where the central one carries the smallest time argument).

The exact equations of motion for dynamic operators may also be written as Langevin equations. We define the corresponding fluctuation operators and derive compact expressions for their correlation functions (CFF). In general the fluctuation operators show no Gaussian behaviour. CFF of second order are δ-correlated and CFF of third order are only δ-correlated with respect to the two highest time arguments.

1. Introduction

In many quantum optical problems a dynamic system (laser modes, characterised by harmonic oscillators, and N-level atoms) is coupled to a dissipative system (see e.g. Paul 1969, Haken 1970). Our task is to calculate correlation functions of dynamic operators (CFD) as well as correlation functions of fluctuation operators of dynamic variables (CFF) using a special model of the dissipative system.

If the interaction between the dynamic and the dissipative system allows a Markovian approximation then CFD with pyramidal time order may be determined using the usual regression theorem (Haken *et al* 1967, Lax 1968, Agarwal 1974, Mollow 1975, Swain 1981). Let $Q(t)$ be any dynamic operator and $Q_\mu(t)$ a complete set of dynamic operators. If the expectation value $\langle Q(t)\rangle$ is given by

$$\langle Q(t)\rangle = \sum_\mu f_\mu(t, t')\langle Q_\mu(t')\rangle \tag{1.1}$$

where $\langle \dots \rangle$ means the quantum statistical average with respect to the whole system, then according to the regression theorem each CFD $\langle M(t', t'')Q(t)N(t', t'')\rangle$ with $t > t' > t''$ may be written as

$$\langle M(t', t'')Q(t)N(t', t'')\rangle = \sum_\mu f_\mu(t, t')\langle M(t', t'')Q_\mu(t')N(t', t'')\rangle \tag{1.2}$$

where $M(t', t'')$, $N(t', t'')$ are aggregates of dynamic operators possessing the maximum

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time argument t' or t''. The regression theorem can be established by showing that the CFD $\langle M(t', t'')Q(t)N(t', t'')\rangle$ and the expectation value $\langle Q(t)\rangle$ obey the same differential equation with respect to t. Successive application of the regression theorem makes it possible to determine any CFD $\langle A_n(t_n)\ldots A_1(t_1)\rangle$ with a pyramidal arrangement of the time arguments $t_1, \ldots, t_n$ $(t_n \leqslant \ldots \leqslant t_s \geqslant t_{s-1} \geqslant \ldots \geqslant t_1;\ n \geqslant s \geqslant 1)$. This procedure breaks down if a CFD contains $m \geqslant 2$ not adjacent operators possessing the maximum time argument.

In order to also cover these situations in this paper we extend the usual regression theorem (Knöll 1983). We prove that any CFD satisfies a differential equation with respect to the maximum time argument. The structure of the differential equation only depends on the m operators possessing the maximal time argument. The other $(n-m)$ operators possessing smaller time arguments only enter in the initial condition at time t', the second in height time argument. Solving these differential equations enables one to calculate successively any CFD, as well as those possessing a non-pyramidal time arragement. However in general, knowledge of the time behaviour of expectation values is not sufficient for determining a CFD with $m \geqslant 2$ operators carrying the maximum time argument.

As an example of a CFD with an unusual, non-pyramidal time arrangement we mention the intensity correlation function of radiation measured by two ideal photodetectors with a frequency filter in front of each (Knöll and Weber 1982). Using Fabry-Perot interferometers with filter functions (Eberly and Wódkiewicz 1977)

$$F(t, t_1, t_2; \omega, \Gamma) = 2\Gamma \exp[-(\Gamma + i\omega)(t - t_2) - (\Gamma - i\omega)(t - t_1)] \tag{1.3}$$

(where Γ is the bandwidth and ω the setting frequency) the measured intensity correlation function is given by ($t \geqslant t'$ assumed)

$$S(\boldsymbol{r}'t'\omega'; \boldsymbol{r}t\omega) = \int_{-\infty}^{t'} dt_2' \int_{-\infty}^{t'} dt_1' \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_1\, F(t', t_1', t_2'; \omega', \Gamma')$$
$$\times F(t, t_1, t_2; \omega, \Gamma)\langle \boldsymbol{E}^{(-)}(\boldsymbol{r}'t_2')\boldsymbol{E}^{(-)}(\boldsymbol{r}t_2)\boldsymbol{E}^{(+)}(\boldsymbol{r}t_1)\boldsymbol{E}^{(+)}(\boldsymbol{r}'t_1')\rangle \tag{1.4}$$

($\boldsymbol{r}, \boldsymbol{r}'$ are the positions of the photodetectors, $\boldsymbol{E}^{(+)}$ ($\boldsymbol{E}^{(-)}$) is the positive (negative) frequency part of the electric field operator $\boldsymbol{E} = \boldsymbol{E}^{(+)} + \boldsymbol{E}^{(-)}$). Because of the filter action the time arguments t_1, t_2 in $\langle \boldsymbol{E}^{(-)}(\boldsymbol{r}'t_2')\boldsymbol{E}^{(-)}(\boldsymbol{r}t_2)\boldsymbol{E}^{(+)}(\boldsymbol{r}t_1)\boldsymbol{E}^{(+)}(\boldsymbol{r}'t_1')\rangle$ may be smaller than t_1', t_2', and a non-pyramidal time arrangement results.

The second problem considered in this paper concerns correlation functions of fluctuation operators (CFF). The Heisenberg equations of motion for dynamic operators can be written as operator Langevin equations containing fluctuation operators, the correlation functions of which are of interest (Sauermann 1965, Paul 1969, Haken 1970, Agarwal 1974, Ponath and Schubert 1977). In the case of a harmonic oscillator as the dynamic system the fluctuation operators show Gaussian behaviour (Paul 1969, Haken 1970), however for an N-level atom as the dynamic system CFF of higher order ($n > 2$) are not known. The assumption of Gaussian behaviour leads to contradictions within the theory. Another assertion was made by Senitzky (1967), but it was not checked by a microscopic model.

Within the scope of our model system we set up operator Langevin equations, define the corresponding fluctuation operators and derive compact expressions for their correlation functions (CFF). From these we deduce rules for calculation and we obtain insight into their time behaviour (number of occurring δ functions, degree of correlation).

The dissipative system used in our considerations contains several independent baths each being composed of many harmonic oscillators with quasi-continuously distributed eigenfrequencies. The baths may be at zero temperature but they can also contain excitations caused by finite temperatures or by coherent excitations. We may regard the coherent excitation as an external time-dependent force acting on the dynamic system. No special assumptions are needed about the dynamic system.

We use as essential methods and assumptions the Feynman disentangling theorem (Feynman 1951, Sauermann 1965, Haken 1970), some special formulae for Bose operators and a well-defined bath approximation. The latter is a Born and Markovian approximation (Agarwal 1974, Grabert 1982) and all results must be interpreted within this scope. In the course of our calculations which represent an extension of the investigations by Sauermann (1965) we derive compact expressions for CFD and CFF which only contain dynamic operators. These operators are equipped with ordering indices, and a direct evaluation of CFD and CFF is very difficult. However it is possible to derive differential equations which may be handled more easily.

In § 2 we define the model system and in § 3 we derive compact expressions for the time-development operator and for the CFD. Readers not so interested in mathematical details may omit § 3 initially. In § 4 differential equations for the CFD are derived concerning the maximum time argument and the generalised regression theorem is formulated. Some simple examples for application are given in appendix 4. Operator Langevin equations for dynamic operators and the corresponding fluctuation operators are defined in § 5, and their correlation functions are studied. In § 6 we summarise some conclusions and discuss possible extensions of our model with respect to the dissipative system.

2. The model system

Our model system is described by the Hamiltonian H:

$$H = H_{\mathrm{D}} + H_{\mathrm{B}} + H_{\mathrm{J}} = H_0 + H_{\mathrm{J}} \tag{2.1}$$

$$H_{\mathrm{B}} = \sum_{\rho} H_{\mathrm{B}}^{(\rho)}, \qquad H_{\mathrm{J}} = \sum_{\rho} \sum_{\alpha} (B^{\rho\alpha} a^{\rho\alpha +} + B^{\rho\alpha +} a^{\rho\alpha}) \tag{2.2}$$

$$H_{\mathrm{B}}^{(\rho)} = \sum_{\lambda} \omega_{\rho\lambda} b_{\rho\lambda}^{+} b_{\rho\lambda}, \qquad B^{\rho\alpha} = \sum_{\lambda} f_{\alpha\rho\lambda} b_{\rho\lambda} \tag{2.3}$$

where H_{D} (H_{B}) is the Hamiltonian of the dynamic (dissipative) system, and H_{J} is the interaction Hamiltonian. Units with $\hbar = 1$ are used throughout.

The dissipative system is composed of independent baths labelled by the index ρ, and each bath consists of many harmonic oscillators (e.g. photons, phonons) with quasi-continuously distributed eigenfrequencies $\omega_{\rho\lambda}$ (λ denotes the oscillators in one bath). The corresponding creation and annihilation operators $b_{\rho\lambda}^{+}$, $b_{\rho\lambda}$ obey the Bose commutation relations

$$[b_{\rho\lambda}^{+}, b_{\rho'\lambda'}^{+}] = [b_{\rho\lambda}, b_{\rho'\lambda'}] = 0, \qquad [b_{\rho\lambda}, b_{\rho'\lambda'}^{+}] = \delta_{\rho\rho'}\delta_{\lambda\lambda'} \tag{2.4}$$

where $\delta_{\rho\rho'}$, $\delta_{\lambda\lambda'}$ are Kronecker symbols. H_{J} is assumed to be linear in $b_{\rho\lambda}$ and $b_{\rho\lambda}^{+}$. The coupling constants $f_{\alpha\rho\lambda}$, the dynamic operators $a^{\rho\alpha}$, $a^{\rho\alpha +}$ and the range of summation for α depend on the special dynamic system under consideration and need not be

specified in our investigation. Defining

$$\hat{Q}(t) = \exp(\mathrm{i}H_0 t) Q \exp(-\mathrm{i}H_0 t) \tag{2.5}$$

where Q is any operator it follows that

$$\hat{b}_{\rho\lambda}(t) = \exp(-\mathrm{i}\omega_{\rho\lambda} t) b_{\rho\lambda}. \tag{2.6}$$

Further we assume

$$\hat{a}^{\rho\alpha}(t) = \exp(-\mathrm{i}\omega_{\rho\alpha} t) a^{\rho\alpha} \tag{2.7}$$

where $\omega_{\rho\alpha}$ are characteristic atomic frequencies. H_J in (2.1)-(2.3) is the most general expression for the interaction linear in $b_{\rho\lambda}$ and $b^+_{\rho\lambda}$. In the following we assume $\omega_{\rho\alpha} \neq \omega_{\rho\alpha'}$ for $\alpha \neq \alpha'$, but we mention that this is not an essential restriction, and it may easily be relaxed. The statistical operator ρ at $t = 0$ is written as

$$\rho = \rho_{\mathrm{D}} \cdot \rho_{\mathrm{B}}, \qquad \rho_{\mathrm{B}} = \prod_{\rho} \rho_{\mathrm{B}}^{(\rho)} \tag{2.8}$$

where $\rho_{\mathrm{D}}(\rho_{\mathrm{B}}^{(\rho)})$ is the statistical operator of the dynamic system (bath (ρ)). In this paper each bath (ρ) is characterised by one of the two operators

$$\rho_{\mathrm{B}}^{(\rho)} = \exp(-H_{\mathrm{B}}^{(\rho)}/kT^{(\rho)}) / \mathrm{Tr}_{\mathrm{B}^{(\rho)}} \exp(-H_{\mathrm{B}}^{(\rho)}/kT^{(\rho)}) \tag{2.9a}$$

$$\rho_{\mathrm{B}}^{(\rho)} = |\{\beta_{\rho\lambda}\}\rangle\langle\{\beta_{\rho\lambda}\}|, \qquad |\{\beta_{\rho\lambda}\}\rangle = \prod_{\lambda} |\beta_{\rho\lambda}\rangle, \qquad b_{\rho\lambda}|\beta_{\rho\lambda}\rangle = \beta_{\rho\lambda}|\beta_{\rho\lambda}\rangle. \tag{2.9b}$$

Equation (2.9a) describes a bath at temperature $T^{(\rho)}$ (k is the Boltzmann constant), and different baths may possess different $T^{(\rho)}$. Equation (2.9b) describes a bath at zero temperature but with coherent excitations characterised by the amplitudes $\beta_{\rho\lambda}$ of the global coherent state $|\{\beta_{\rho\lambda}\}\rangle$. It acts on the dynamic system both as a bath of temperature zero and as an external time-dependent force characterised by the $\beta_{\rho\lambda}$.

One aim of this paper is the investigation of correlation functions of dynamic operators (CFD)

$$\langle A_n(t_n) \ldots A_1(t_1)\rangle_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{B}}(\rho_{\mathrm{B}} A_n(t_n) \ldots A_1(t_1)) \tag{2.10}$$

$$A_i(t_i) = U^+(t_i) A_i U(t_i) \equiv \exp(\mathrm{i}Ht_i) A_i \exp(-\mathrm{i}Ht_i) \tag{2.11}$$

and the derivation of differential equations for their determination. The $A_i(t_i)$ are dynamic operators (not necessarily different) in the full Heisenberg picture. The index i also characterises the position in the CFD. The time arguments t_i are arbitrary, and some of them may be equal. The trace only has to be taken with the states of the dissipative system, and the CFD are still operators in the Hilbert space of the dynamic system. Two adjacent operators with equal time arguments ($t_i = t_{i-1}$) may be considered as one operator:

$$A_i(t_i) \cdot A_{i-1}(t_i) = (A_i \cdot A_{i-1})(t_i).$$

As a concrete example for our model system we mention an N-level atom (dynamic system) interacting with the quantised radiation field (dissipative system). Coherent excitations of the bath may be considered to be caused by an external ideal laser acting on the atom and producing non-linear effects (e.g. resonance fluorescence (Mollow 1975, Kimble and Mandel 1976) and Raman scattering (Agarwal and Jha 1979)).

3. Compact expressions for CFD

3.1. Investigation of the time-development operator U(t)

We start with the time-development operator $U(t) \equiv \exp(-\mathrm{i}Ht)$ which may be written as follows

$$U(t) = \exp(-\mathrm{i}H_0 t)S(t), \qquad S(t) = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau \hat{H}_{\mathrm{J}\tau}(\tau)\right) \tag{3.1}$$

$$\hat{H}_{\mathrm{J}\tau}(\tau) = \sum_{\rho\alpha} (\hat{B}_\tau^{\rho\alpha}(\tau)\hat{a}_\tau^{\rho\alpha+}(\tau) + \hat{B}_\tau^{\rho\alpha+}(\tau)\hat{a}_\tau^{\rho\alpha}(\tau)). \tag{3.2}$$

The index τ in $\hat{H}_{\mathrm{J}\tau}(\tau)$, $\hat{B}_\tau^{\rho\alpha+}(\tau)$, $\hat{a}_\tau^{\rho\alpha+}(\tau)$ ... has the meaning of an ordering parameter. Operators with smaller ordering indices always act before operators with larger ordering indices, and operators with ordering indices may be handled as c-numbers by carrying out formal manipulations (Feynman 1951). In order not to overburden the formulae we shall drop the indices ρ and α and the corresponding summations in § 3. This does not lead to any restriction and later we shall reintroduce the indices ρ and α and the summations. We apply the disentangling theorem of Feynman (1951) on (3.1) in order to separate the b_λ^+ from the b_λ in the exponent. Taking into account the operator identity (where x is any c-number)

$$\exp(xb_\lambda^+)b_{\lambda'}\exp(-xb_\lambda^+) = b_{\lambda'} - x\delta_{\lambda\lambda'}, \tag{3.3}$$

after a transformation of integration variables we get

$$S(t) = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau\, \hat{B}^+(\tau)\hat{a}_\tau(\tau)\right)\exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau \hat{B}(\tau)\hat{a}_\tau^+(\tau)\right)$$
$$\times\exp\left[-\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \left(\sum_\lambda f_\lambda f_\lambda^* \exp[\mathrm{i}(\omega-\omega_\lambda)\tau']\right)\hat{a}_\tau^+(\tau)\hat{a}_{\tau-\tau'}(\tau)\right]. \tag{3.4}$$

We now make a bath approximation using the previously mentioned assumption about the distribution of the eigenfrequencies ω_λ. If $(f_\lambda f_\lambda^*)$ is a smooth function of λ, the sum $\Sigma_\lambda f_\lambda f_\lambda^* \exp[\mathrm{i}(\omega-\omega_\lambda)\tau')]$ should be not equal to zero only for $\tau' < \tau_{\mathrm{B}}$, where τ_{B} is a bath relaxation time assumed to be much smaller than any characteristic time of the dynamic system. For $t \gg \tau_{\mathrm{B}}$ we approximate $\hat{a}_{\tau-\tau'}(\tau)$ in the double integral in (3.4) by $\hat{a}_\tau(\tau)$ and we replace the integration limit τ by ∞. Then the double integral in (3.4) may be replaced by

$$\int_0^t \mathrm{d}\tau\, \gamma\hat{a}_\tau^+(\tau)\hat{a}_\tau(\tau), \qquad \gamma \equiv \int_0^\infty \mathrm{d}\tau\, \langle[\hat{B}(\tau), \hat{B}^+(0)]\rangle_{\mathrm{B}} \exp(\mathrm{i}\omega\tau). \tag{3.5}$$

Such bath approximations will also be used in the further calculations. They are of Born and Markovian nature, and all the following results should be interpreted within this scope.

A repeated application of the Feynman disentangling theorem finally gives

$$U(t) = V_0(t)V_{\mathrm{J}}(t), \qquad V_0(t) = \exp[(-\mathrm{i}H_0 - R)t] \tag{3.6}$$

$$V_{\mathrm{J}}(t) = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau\, \hat{B}^+(\tau)\check{a}_\tau(\tau)\right)\exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau\, \hat{B}(\tau)\check{a}_\tau^+(\tau)\right) \tag{3.7}$$

where

$$R = \gamma a^+ a \qquad \text{and} \qquad \check{Q}(\tau) = V_0^{-1}(\tau)QV_0(\tau) \tag{3.8}$$

for any operator Q. In the expression (3.7) already given by Sauermann (1965) the b_λ^+ and b_λ are normally ordered and free of ordering indices.

3.2. *Investigation of CFD*

Using $U(t)$ from (3.6)-(3.8) we get

$$\langle A_n(t_n)\dots A_1(t_1)\rangle_B = \mathrm{Tr}_B(\rho_B V_J^+(t_n)V_0^+(t_n)A_n V_0(t_n)V_J(t_n)\dots V_J^+(t_1)V_0^+(t_1)A_1 V_0(t_1)V_J(t_1)) \tag{3.9}$$

where in V_J^+ a reverse ordering prescription must be used. Our aim is to arrive at a normal order with respect to b_λ^+ and b_λ in the whole expression in (3.9). After some transformations (see appendix 1) we get

$$\langle A_n(t_n)\dots A_1(t_1)\rangle_B = \mathscr{B}\exp(\psi^0)\exp(\chi^0)\tilde{A}_n(t_n)\dots\tilde{A}_1(t_1) \tag{3.10}$$

$$\mathscr{B} = \left\langle \exp\left(\mathrm{i}\sum_{i=1}^{n}\int_0^{t_i} \mathrm{d}\tau\, \hat{B}^+(\tau)\mathscr{M}_i(\tau)\right)\exp\left(-\mathrm{i}\sum_{i=1}^{n}\int_0^{t_i}\mathrm{d}\tau\,\hat{B}(\tau)\mathscr{N}_i(\tau)\right)\right\rangle_B \tag{3.11}$$

$$\chi^0 = \sum_{i=1}^{n}\int_0^{t_i}\mathrm{d}\tau(\gamma+\gamma^*)(\check{a}_{i\bar{\tau}}(\tau))^+\check{a}_{i\tau}(\tau) \tag{3.12}$$

$$\psi^0 = \sum_{i>j}\int_0^{t_{ij}}\mathrm{d}\tau\,\mathscr{L}_{ij}^0(\tau), \qquad t_{ij}=\min\{t_i,t_j\} \tag{3.13}$$

$$\mathscr{L}_{ij}^0(\tau) = (\gamma+\gamma^*)\mathscr{N}_i(\tau)\mathscr{M}_j(\tau) \tag{3.14}$$

$$\mathscr{M}_i(\tau) = (\check{a}_{i\bar{\tau}}^+(\tau))^+ - \check{a}_{i\tau}(\tau) \tag{3.15}$$

$$\mathscr{N}_i(\tau) = \check{a}_{i\tau}^+(\tau) - (\check{a}_{i\bar{\tau}}(\tau))^+ \tag{3.16}$$

$$\tilde{Q}(\tau) = V_0^+(\tau)QV_0(\tau) \qquad \text{for any operator } Q. \tag{3.17}$$

Here the following ordering prescriptions must be used. Concerning the index i all operators have to be ordered from right to left in increasing sequence. Operators $a_{i\tau}$, $a_{i\tau}^+$ ($a_{i\bar{\tau}}$, $a_{i\bar{\tau}}^+$) are arranged to the right (left) from A_i. Operators $a_{i\tau}$, $a_{i\tau}^+$ ($a_{i\bar{\tau}}$, $a_{i\bar{\tau}}^+$) with equal i but different τ are arranged from right to left (left to right) with increasing τ. We call attention to the appearence of the time arguments $t_{ij}=\min\{t_i,t_j\}$ which express a bath-mediated correlation between A_i and A_j.

Next we calculate $\mathscr{B}$ using equation (2.9) for ρ_B. Since the b_λ^+, b_λ are normally ordered the evaluation is very easily performed in the case of (2.9b), and for the case of (2.9a) we use the following formula for Bose operators b, b^+ (where x, y are any c-numbers)

$$\mathrm{Tr}_B[\exp(xb^+)\exp(yb)\exp(-\omega b^+b/kT)]/\mathrm{Tr}_B\exp(-\omega b^+b/kT) = \exp(xy\bar{n})$$
$$\bar{n} = [\exp(\omega/kT)-1]^{-1}. \tag{3.18}$$

The results may now be summarised as follows

$$\langle A_n(t_n)\dots A_1(t_1)\rangle_B = \exp(\psi)\exp(\chi)\tilde{A}_n(t_n)\dots\tilde{A}_1(t_1) \tag{3.19}$$

$$\chi = \chi^0+\chi^T, \qquad \psi = \psi^0+\psi^T \tag{3.20a}$$

$$\chi = \chi^0+\chi^c, \qquad \psi = \psi^0 \tag{3.20b}$$

$$\chi^c = \mathrm{i}\sum_{i=1}^{n}\int_0^{t_i} \mathrm{d}\tau(\beta^*(\tau)\mathcal{M}_i(\tau) - \beta(\tau)\mathcal{N}_i(\tau)) \tag{3.21}$$

$$\chi^T = \sum_{i=1}^{n}\int_0^{t_i} \mathrm{d}\tau(\gamma^{T*}\mathcal{M}_i(\tau)\mathcal{N}_i(\tau^-) + \gamma^T\mathcal{M}_i(\tau^-)\mathcal{N}_i(\tau)) \tag{3.22}$$

$$\psi^T = \sum_{i\neq j}\int_0^{t_{ij}} \mathrm{d}\tau\,\mathcal{L}_{ij}^T(\tau), \qquad \mathcal{L}_{ij}^T(\tau) = (\gamma^T + \gamma^{T*})\mathcal{M}_i(\tau)\mathcal{N}_j(\tau) \tag{3.23}$$

$$\gamma^T = \int_0^{\infty} \mathrm{d}\tau\langle\hat{B}^+(0)\hat{B}(\tau)\rangle_{\mathrm{B}}\exp(\mathrm{i}\omega\tau) \tag{3.24a}$$

$$\beta(\tau) = \sum_{\lambda} f_\lambda\beta_\lambda\exp(-\mathrm{i}\omega_\lambda\tau). \tag{3.24b}$$

The two cases (a) and (b) in (3.20) and (3.24) correspond to the two cases of the bath in (2.9). The argument τ^- in (3.22) should be treated as being different from τ only with respect to the ordering of the operators in the sense $\tau^- < \tau$. After ordering the operators, τ^- should be set equal to τ.

Equation (3.19) is the compact expression for the CFD we seek. It depends only on dynamic operators which are equipped with ordering indices. The special properties of the dissipative system enter into the constants γ, γ^T and β_λ only.

The ordering prescriptions range over all operators on the RHS of equation (3.19). A direct computation is therefore very cumbersome. However, as we shall see, it is possible to derive differential equations from (3.19) with respect to the maximum time argument which may be handled more easily than the compact expression in equation (3.19).

4. Generalised regression theorem

We define $t = \max\{t_n, \ldots, t_1\}$ and we assume that m operators labelled by $A_{i_m} = Q_m, \ldots, A_{i_1} = Q_1 (n \geqslant i_m > \ldots > i_1 \geqslant 1)$ possess this maximum time argument $t (t_{i_m} = \ldots = t_{i_1} = t)$. The corresponding CFD is now written as

$$\langle A_n(t_n)\ldots A_1(t_1)\rangle_{\mathrm{B}} = \langle\!\langle Q_m(t);\ldots;Q_1(t)\rangle\!\rangle \tag{4.1}$$

where in $\langle\!\langle Q_m(t);\ldots;Q_1(t)\rangle\!\rangle$ only the operators with the maximum time argument t are explicitly mentioned. Differentiating (4.1) with respect to t and using (3.19)–(3.24) and (3.12)–(3.17) we obtain (for details see appendix 2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle Q_m(t);\ldots;Q_1(t)\rangle\!\rangle$$

$$= \sum_{a=1}^{m}\langle\!\langle Q_m(t);\ldots;(\mathcal{R}Q_a)(t);\ldots;Q_1(t)\rangle\!\rangle + \sum_{a>b}\mathcal{R}_{ab}\langle\!\langle Q_m(t);\ldots;Q_1(t)\rangle\!\rangle \tag{4.2}$$

where

$$\mathcal{R}Q_a = \mathrm{i}[H_{\mathrm{D}}', Q_a] + \sum_{\rho\alpha} 2\mu_{\rho\alpha}(a^{\rho\alpha+}Q_a a^{\rho\alpha} - \tfrac{1}{2}[a^{\rho\alpha+}a^{\rho\alpha}, Q_a]_+) + \sum_{\rho\alpha} 2\mu_{\rho\alpha}^T(a^{\rho\alpha}Q_a a^{\rho\alpha+} - \tfrac{1}{2}[a^{\rho\alpha}a^{\rho\alpha+}, Q_a]_+) \tag{4.3}$$

$$\mathcal{R}_{ab}\langle\langle Q_m(t);\ldots;Q_1(t)\rangle\rangle$$
$$=\sum_{\rho\alpha}2\mu_{\rho\alpha}\langle\langle Q_m(t);\ldots;[Q_a,a^{\rho\alpha+}](t);\ldots;[a^{\rho\alpha},Q_b](t);\ldots;Q_1(t)\rangle\rangle$$
$$+\sum_{\rho\alpha}2\mu^T_{\rho\alpha}\langle\langle Q_m(t);\ldots;[Q_a,a^{\rho\alpha}](t);\ldots;[a^{\rho\alpha+},Q_b](t);\ldots;Q_1(t)\rangle\rangle \tag{4.4}$$

and

$$H'_{\mathrm{D}}=H_{\mathrm{D}}+\sum_{\rho\alpha}(v_{\rho\alpha}a^{\rho\alpha+}a^{\rho\alpha}-v^T_{\rho\alpha}a^{\rho\alpha}a^{\rho\alpha+})+\sum_{\rho\alpha}{}'(\beta^{\rho\alpha}(t)a^{\rho\alpha+}+\beta^{\rho\alpha}{}^*(t)a^{\rho\alpha}). \tag{4.5}$$

The summations over ρ and α are now reintroduced. $[\cdot,\cdot]_+$ is the anticommutator, and the real constants $\mu_{\rho\alpha}$, $\mu^T_{\rho\alpha}$, $v_{\rho\alpha}$ and $v^T_{\rho\alpha}$ are defined by

$$\mu_{\rho\alpha}+\mathrm{i}v_{\rho\alpha}=\int_0^\infty \mathrm{d}\tau\langle\hat{B}^{\rho\alpha}(\tau)\hat{B}^{\rho\alpha+}(0)\rangle_{\mathrm{B}}\exp(\mathrm{i}\omega_{\rho\alpha}\tau)$$
$$\mu^T_{\rho\alpha}+\mathrm{i}v^T_{\rho\alpha}=\int_0^\infty \mathrm{d}\tau\langle\hat{B}^{\rho\alpha+}(0)\hat{B}^{\rho\alpha}(\tau)\rangle_{\mathrm{B}}\exp(\mathrm{i}\omega_{\rho\alpha}\tau) \tag{4.6a}$$

for $\rho^{(\rho)}_{\mathrm{B}}$ given by (2.9*a*), and in the case of (2.9*b*) we have

$$\mu_{\rho\alpha}+\mathrm{i}v_{\rho\alpha}=\int_0^\infty \mathrm{d}\tau\langle[\hat{B}^{\rho\alpha}(\tau),\hat{B}^{\rho\alpha+}(0)]\rangle_{\mathrm{B}}\exp(\mathrm{i}\omega_{\rho\alpha}\tau)$$
$$\mu^T_{\rho\alpha}=v^T_{\rho\alpha}=0,\qquad \beta^{\rho\alpha}(t)=\sum_\lambda f_{\alpha\rho\lambda}\beta_{\rho\lambda}\exp(-\mathrm{i}\omega_{\rho\lambda}t). \tag{4.6b}$$

For $\beta^{\rho\alpha}(t)\equiv 0$ the superoperator $\mathcal{R}$ in (4.3) acting on dynamic operators has the form of a generator of a quantum dynamic semigroup (Lindblad 1976).

The Hamiltonian part H'_{D} of $\mathcal{R}$ contains in addition to H_{D} a level shift term and a term describing the coherent excitations in the baths (the sum $\Sigma'_{\rho\alpha}$ goes only over ρ with $\rho^{(\rho)}_{\mathrm{B}}$ given by (2.9*b*)). Equation (4.2) together with (4.3)-(4.6) represents the essential contents of the generalised regression theorem. It is seen that the structure of the differential equation (4.2) for $\langle\langle Q_m(t);\ldots;Q_1(t)\rangle\rangle\equiv\langle A_n(t_n)\ldots A_1(t)\rangle_{\mathrm{B}}$ is only determined by the operators $Q_m(t)\equiv A_{i_m}(t),\ldots,\ Q_1(t)\equiv A_{i_1}(t)$ possessing the maximum time argument t. The other $(n-m)$ operators possessing smaller time arguments are only essential for the initial condition. This behaviour is due to the Markovian approximation used in deriving the differential equation (4.2).

Equation (4.2) must be solved under the initial condition

$$\langle\langle Q_m(t);\ldots;Q_1(t)\rangle\rangle_{t=t'}=\langle\langle Q_m(t');\ldots;Q_1(t')\rangle\rangle,$$

where t' is the second in height time argument. The solution of (4.2) depends therefore on CFD with the maximum time argument t' and further time arguments occurring in the original CFD and being smaller than t and t'.

To illustrate the differential equation (4.2) in more detail we consider the special cases $m=1$ and $m=2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle Q_1(t)\rangle\rangle=\langle\langle(\mathcal{R}Q_1)(t)\rangle\rangle \tag{4.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle Q_2(t);\, Q_1(t)\rangle\rangle$$

$$=\langle\langle(\mathcal{R}Q_2)(t);\, Q_1(t)\rangle\rangle+\langle\langle Q_2(t);\,(\mathcal{R}Q_1)(t)\rangle\rangle+\mathcal{R}_{21}\langle\langle Q_2(t);\, Q_1(t)\rangle\rangle. \tag{4.8}$$

Equation (4.7) also comprises the case $n=1$ in which $\langle\langle Q_1(t)\rangle\rangle$ is identical with an expectation value: $\langle\langle Q_1(t)\rangle\rangle=\langle Q_1(t)\rangle_{\mathrm{B}}$. This means that a CFD $\langle A_n(t_n)\ldots A_1(t_1)\rangle_{\mathrm{B}}$ with only one operator possessing the maximum time argument t and the expectation value of this operator obey the same differential equation. This is the content of the well known usual regression theorem which is also contained in our general result.

We would like to mention that after taking the expectation value over the dynamic system, the differential equation may also be derived for $\langle Q_1(t)\rangle$ by using an equation of motion for the density matrix (Haken 1970) and passing over to the expectation value of Q_1. In that case the dissipative system is not restricted to be composed of harmonic oscillators, and the $B^{\rho\alpha}$ may be general bath operators.

Equation (4.8) shows that because of the term $\mathcal{R}_{21}\langle\langle Q_2(t);\, Q_1(t)\rangle\rangle$ the solution for the expectation values is not sufficient to determine the time behaviour of $\langle\langle Q_2(t);\, Q_1(t)\rangle\rangle$ (and in general of $\langle\langle Q_m(t);\ldots;\, Q_1(t)\rangle\rangle$ as can be seen from (4.2)). The differential equation (4.8) must be solved separately. Only in special cases ($Q_2=1$ or $Q_1=1$ or Q_2 and Q_1 adjacent) does equation (4.8) reduce to (4.7) (with Q_1 or Q_2 or $Q_2\cdot Q_1$ respectively). On the other hand equation (4.8) (and in general (4.2)) does not contain more information about the dissipative system (expressed by $\mu_{\rho\alpha}$, $v_{\rho\alpha}$, $\mu^T_{\rho\alpha}$, $v^T_{\rho\alpha}$, $\beta^{\rho\alpha}(t)$) than equation (4.7) for expectation values.

In concluding this section we state that CFD with any time arrangements can be determined by successively solving differential equations of the general type (4.2). In appendix 4 some simple examples are given for the dynamic system being a harmonic oscillator and a two-level atom.

5. Correlation functions of fluctuation operators (CFF)

In the case of a dynamic system interacting with a dissipative system the equations of motion for dynamic operators $A(t)$ can formally be written as operator Langevin equations (Sauermann 1965, Paul 1969, Haken 1970, Agarwal 1974). We write

$$(\mathrm{d}/\mathrm{d}t)A(t)=(\mathcal{R}A)(t)+F_A(t);\qquad F_A(t)\equiv(\mathrm{d}/\mathrm{d}t)A(t)-(\mathcal{R}A)(t),$$
$$\langle F_A(t)\rangle_{\mathrm{B}}=0 \tag{5.1}$$

where $F_A(t)$ are the corresponding fluctuation operators. We are interested in the correlation functions of these fluctuation operators (CFF), the properties of which are not yet fully known. In § 5.1 we derive compact expressions for CFF and study their time behaviour. More detailed expressions for CFF up to fourth order are given in § 5.2.

5.1. Compact expressions for CFF

We denote a CFF by $K(A_n,\ldots,A_1;\, t_n,\ldots t_1)$:

$$K(A_n,\ldots A_1;\, t_n,\ldots,t_1)\equiv\langle F_{A_n}(t_n)\ldots(F_{A_1}(t_1)\rangle_{\mathrm{B}}. \tag{5.2}$$

After some identical transformations (see appendix 3) we get the compact expression

$$K(A_n, \ldots, A_1; t_n, \ldots, t_1)$$
$$= [(\partial/\partial t_n) \ldots (\partial/\partial t_1) \exp(\psi)] \exp(\chi) \tilde{A}_n(t_n) \ldots \tilde{A}_1(t_1) \tag{5.3}$$

in which differentiations of ψ have still to be carried out. $\tilde{A}_i(t_i)$, ψ and χ are defined in § 3.2, and ψ depends only on the time arguments $t_{ij} = \min\{t_i, t_j\}$. From (5.3) we may conclude immediately that each CFF is zero if one of the time arguments $t_1, \ldots, t_n$ is larger than all the others. However if one time argument is smaller than all the others, an expression not equal to zero may result.

Taking into account

$$t_{ij} = \theta(t_i - t_j) t_j + \theta(t_j - t_i) t_i, \qquad (\partial/\partial t_i) t_{ij} = (\partial/\partial t_i) t_{ji} = \theta(t_j - t_i) \tag{5.4}$$

where $\theta(x)$ is the step function ($\theta(x) = 1(0)$ for $x > 0 (x < 0)$) and $\delta(x) = \mathrm{d}\theta(x)/\mathrm{d}x$ is the Dirac δ function we obtain

$$\frac{\partial}{\partial t_i} \exp(\psi) = \sum_j^{j \neq i} \theta(t_j - t_i) \mathscr{R}_{ji}^{t_i} \exp(\psi) \tag{5.5}$$

$$\frac{\partial}{\partial t_j} \theta(t_j - t_i) \mathscr{R}_{ji}^{t_i} \exp(\psi)$$
$$= \delta(t_j - t_i) \mathscr{R}_{ji}^{t_i} \exp(\psi) + \sum_m^{m \neq j} \theta(t_m - t_j) \theta(t_j - t_i) \mathscr{R}_{mj}^{t_i} \mathscr{R}_{ji}^{t_i} \exp(\psi) \tag{5.6}$$

where

$$\mathscr{R}_{ji}^{t_i} = \mathscr{L}_{ji}^0(t_i) + \mathscr{L}_{ji}^T(t_i) + \mathscr{L}_{ij}^T(t_i) \qquad \text{for } j > i \tag{5.7}$$

$$\mathscr{R}_{ji}^{t_i} = \mathscr{R}_{ij}^{t_i} \qquad \text{for } j < i \tag{5.8}$$

and $\mathscr{L}_{ij}^0(t)$ and $\mathscr{L}_{ij}^T(t)$ are the expressions as defined in § 3.2. The action of the superoperator $\mathscr{R}_{ji}^{t_i}$ on any CFD $\langle A_n(t_n) \ldots A_1(t_1) \rangle_B$ may also be explained as follows: $\langle A_n(t_n) \ldots A_1(t_1) \rangle_B$ has to be calculated step by step with the help of the generalised regression theorem but with the initial condition $\mathscr{R}_{ji} \langle A_n(t_n) \ldots A_1(t_1) \rangle_B$ at time t_i (cf equation (5.12)) where the action of $\mathscr{R}_{ji}$ is explained by equation (4.4).

Carrying out the differentiations further, the following structure of the CFF may be recognised. Each CFF consists of several terms, and each term is proportional to a product of k δ functions and $(n-2k)$ θ functions, where $1 \leqslant k \leqslant [n/2]$ and $[n/2]$ being the largest integer less than or equal to $n/2$. In each term any time argument t_i appears in the product of the δ and the θ functions once at most in the δ functions, but it can appear in several θ functions.

From this structure of the CFF we may deduce that in general the $F_A(t)$ show no Gaussian behaviour (an exception is given by the harmonic oscillator as a dynamic system linearly coupled to the dissipative system). Our result does not confirm the assertion of Senitzky (1967) according to which for an atom as the dynamic system a CFF of order n should contain a term proportional to $\delta(t_1 - t_2)\delta(t_1 - t_3) \ldots \delta(t_1 - t_n)$. Terms with such a strong correlation are absent in our general expression (5.3).

5.2. CFF up to fourth order

In the case of second order from (5.3), (5.5), (5.6) we obtain directly

$$K(A_2, A_1; t_2, t_1) \equiv \langle F_{A_2}(t_2) F_{A_1}(t_1) \rangle_B = \delta(t_2 - t_1) \mathscr{R}_{21} \langle A_2(t_1) A_1(t_1) \rangle_B \tag{5.9}$$

which may also be written in the form

$$K(A_2, A_1; t_2, t_1)$$
$$= \delta(t_2 - t_1)[((\mathcal{R}(A_2 \cdot A_1))(t_1))_B - \langle(\mathcal{R}A_2)(t_1) \cdot A_1(t_1)\rangle_B$$
$$- \langle A_2(t_1) \cdot (\mathcal{R}A_1)(t_1)\rangle_B]. \tag{5.10}$$

CFF of second order are always δ-correlated in the given model. The proportionality factor is a bath expectation value of dynamic operators and may be time dependent. Equation (5.9) is also valid for baths with coherent excitations representing external time-dependent forces acting on the dynamic system. In special cases the evaluation of (5.9) gives results in agreement with corresponding formulae e.g. by Sauermann (1965) and Ponath and Schubert (1977).

In third order from the evaluation of (5.3) it follows that

$$K(A_3, A_2, A_1; t_3, t_2, t_1)$$
$$\equiv \langle F_{A_3}(t_3) F_{A_2}(t_2) F_{A_1}(t_1)\rangle_B$$
$$= \delta(t_3 - t_2)\theta(t_2 - t_1)(\mathcal{R}_{31}^{t_1} + \mathcal{R}_{21}^{t_1})\mathcal{R}_{32}^{t_2}\langle A_3(t_2)A_2(t_2)A_1(t_1)\rangle_B$$
$$+ \delta(t_2 - t_1)\theta(t_1 - t_3)(\mathcal{R}_{32}^{t_3} + \mathcal{R}_{31}^{t_3})\mathcal{R}_{21}^{t_1}\langle A_3(t_3)A_2(t_1)A_1(t_1)\rangle_B$$
$$+ \delta(t_1 - t_3)\theta(t_3 - t_2)(\mathcal{R}_{32}^{t_2} + \mathcal{R}_{21}^{t_2})\mathcal{R}_{31}^{t_3}\langle A_3(t_3)A_2(t_2)A_1(t_3)\rangle_B. \tag{5.11}$$

This means that the calculation of $K(A_3, A_2, A_1; t_3, t_2, t_1)$ requires the evaluation of CFD of second or third order but with only two different time arguments. For example $\mathcal{R}_{32}^{t_2}\mathcal{R}_{31}^{t_3}\langle A_3(t_3)A_2(t_2)A_1(t_3)\rangle_B$ with $t_3 > t_2$ has to be determined by the solution of the differential equation for $(\mathcal{R}_{31}^{t_3}\langle\langle A_3(t_3); A_1(t_3)\rangle\rangle)$ with the initial condition

$$(\mathcal{R}_{31}^{t_3}\langle\langle A_3(t_3); A_1(t_3)\rangle\rangle)_{t_3 = t_2} = \mathcal{R}_{32}\mathcal{R}_{31}\langle A_3(t_2)A_2(t_2)A_1(t_2)\rangle_B. \tag{5.12}$$

CFF of fourth order contain many terms which may be divided into four classes according to their time dependence, one term of each class (K_a, K_b, K_c, K_d) is presented:

$$K_a = \delta(t_1 - t_2)\delta(t_3 - t_4)\mathcal{R}_{43}^{t_3}\mathcal{R}_{21}^{t_2}\langle A_4(t_3)A_3(t_3)A_2(t_2)A_1(t_2)\rangle_B$$
$$K_b = \delta(t_1 - t_2)\theta(t_2 - t_3)\theta(t_3 - t_4)\mathcal{R}_{43}^{t_4}\mathcal{R}_{32}^{t_3}\mathcal{R}_{21}^{t_2}\langle A_4(t_4)A_3(t_3)A_2(t_2)A_1(t_2)\rangle_B$$
$$K_c = \delta(t_1 - t_2)\theta(t_2 - t_3)\theta(t_2 - t_4)\mathcal{R}_{42}^{t_4}\mathcal{R}_{32}^{t_3}\mathcal{R}_{21}^{t_2}\langle A_4(t_4)A_3(t_3)A_2(t_2)A_1(t_2)\rangle_B \tag{5.13}$$
$$K_d = \delta(t_1 - t_2)\theta(t_2 - t_4)\theta(t_1 - t_3)\mathcal{R}_{31}^{t_3}\mathcal{R}_{42}^{t_4}\mathcal{R}_{21}^{t_2}\langle A_4(t_4)A_3(t_3)A_2(t_2)A_1(t_2)\rangle_B.$$

In general each CFF of order n contains CFD with l different time arguments ($n - [n/2] \leq l \leq n - 1$). These CFD must be determined before the corresponding CFF can be calculated.

6. Summary

In this paper we derived rules for calculating CFD (generalised regression theorem which is an extension of the well known usual regression theorem) and CFF of any order and with any time arrangement but without making special assumptions about the dynamic system. The main equations are given by (4.2) and (5.3) together with the corresponding definitions. These results have been derived under the assumption

of a special dissipative system. Furthermore a bath approximation has been made which may be classified as a Born and Markovian approximation.

The results suggest arguments for a further generalisation. We expect also that for more general dissipative systems but with ρ_B given by (2.9*a*) in a Born and Markovian approximation the differential equation (4.2) in connection with the relations (2.1), (2.2), and (4.3)-(4.6*a*) should be used for CFD, where the $B^{\rho\alpha}$ are now more general bath operators. Then the generalised regression theorem may be applied also in the case of an arbitrary dissipative system. This suggestion is supported by the following fact. It is possible to derive the differential equation (4.2) in another way, applicable for general dissipative systems (Schubert 1983). One method consists in expanding the CFD in powers of H_J and making approximations similar to those used in deriving density-matrix equations (Haken 1970). However now additional factorisation assumptions are necessary concerning bath and dynamic variables.

Since completing our calculations we have seen a paper by von Waldenfels (1982) in which the problem of light emission and absorption of a two-level atom is treated in the frame of quantum stochastic processes. This approach enables him to calculate correlation functions of atomic operators with any time arrangement. It seems to us that his results correspond to our formulae if we apply them to the model of a two-level atom interacting with the quantised radiation field as a dissipative system.

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Appendix 1

In order to arrive at a normal order with respect to b_λ^+ and b_λ in (3.9) we have to transfer all exponential functions with arguments $B(B^+)$ to the right (left). In this procedure the dynamic operators $\check{a}_\tau(\tau), \ldots$ may be handled as c-numbers if we equip them with an additional ordering index i indicating the origin from $V_J(t_i)$ or $V_J^+(t_i)$. Using the operator identity

$$\exp(P)\exp(Q) = \exp(Q)\exp(P)\exp([P, Q]) \tag{A1.1}$$

if $[P, Q]$ is a c-number we obtain

$$\begin{aligned}\exp\left(-\mathrm{i}\int_0^{t_i} \mathrm{d}\tau_i\, \hat{B}(\tau_i)\check{a}_{i\tau_i}^+(\tau_i)\right)\exp\left(-\mathrm{i}\int_0^{t_j} \mathrm{d}\tau_j\, \hat{B}^+(\tau_j)\check{a}_{j\tau_j}(\tau_j)\right)\\ = \exp\left(-\mathrm{i}\int_0^{t_j} \mathrm{d}\tau_j\, \hat{B}^+(\tau_j)\check{a}_{j\tau_j}(\tau_j)\right)\exp\left(-\mathrm{i}\int_0^{t_i} \mathrm{d}\tau_i\, \hat{B}(\tau_i)\check{a}_{i\tau_i}^+(\tau_i)\right)\\ \times\exp\left(-\int_0^{t_i} \mathrm{d}\tau_i\int_0^{t_j} \mathrm{d}\tau_j[\hat{B}(\tau_i), \hat{B}^+(\tau_j)]\check{a}_{i\tau_i}^+(\tau_i)\check{a}_{j\tau_j}(\tau_j)\right).\end{aligned} \tag{A1.2}$$

Using the bath approximation explained earlier the double integral $D = \int_0^{t_i} \mathrm{d}\tau_i \int_0^{t_j} \mathrm{d}\tau_j \ldots$

in (A1.2) may be approximated by

$$D = \int_0^{t_{ij}} \mathrm{d}\tau(\gamma+\gamma^*)\check{a}_{i\tau}^+(\tau)\check{a}_{j\tau}(\tau), \qquad t_{ij} = \min\{t_i, t_j\} \tag{A1.3}$$

because of the small bath relaxation times τ_B. Continuing in this way we arrive at equation (3.10).

Appendix 2

In order to derive equation (4.2) by differentiating (4.1) with respect to t we first consider the case $m = 1$. We now label the largest time argument t_i by t and set $A_i = Q$. Differentiating (3.19) with respect to t gives

$$\begin{aligned}(\mathrm{d}/\mathrm{d}t)&\langle\!\langle Q(t)\rangle\!\rangle \\ &= (\mathrm{d}/\mathrm{d}t)\langle A_n(t_n)\ldots A_i(t)\ldots A_1(t_1)\rangle_B \\ &= \exp(\psi)[(\mathrm{d}/\mathrm{d}t)\exp(\chi)]\tilde{A}_n(t_n)\ldots\tilde{A}_i(t)\ldots\tilde{A}_1(t_1) \\ &\quad + \exp(\psi)\exp(\chi)\tilde{A}_n(t_n)\ldots[(\mathrm{d}/\mathrm{d}t)\tilde{A}_i(t)]\ldots\tilde{A}_1(t_1).\end{aligned} \tag{A2.1}$$

From (3.6), (3.17) we have

$$(\mathrm{d}/\mathrm{d}t)\tilde{A}_i(t) = V_0^+(t)\{\mathrm{i}[H_0, A_i] - (RA_i + A_iR)\}V_0(t) \tag{A2.2}$$

and from the first term in equation (A2.1) we get

$$\exp(\psi)\exp(\chi)(\tilde{A}_n(t_n)\ldots[(\gamma+\gamma^*)(\check{a}_{i\bar{t}}(t))^+\check{a}_{it}(t)\tilde{A}_i(t)]\ldots\tilde{A}_1(t_1)) + \ldots \tag{A2.3}$$

where the term explicitly written down results from χ^0 and the further contributions are due to χ^T or χ^c. Since t is the maximum time argument, from the ordering prescription and equation (3.17) we obtain

$$\begin{aligned}(\check{a}_{i\bar{t}}(t))^+\check{a}_{it}(t)\tilde{A}_i(t) &= (\check{a}(t))^+\tilde{A}_i(t)\check{a}(t) \\ &= V_0^+(t)a^+A_iaV_0(t) = \widetilde{(a^+A_ia)}(t).\end{aligned} \tag{A2.4}$$

Combining all partial results we arrive at

$$(\mathrm{d}/\mathrm{d}t)\langle\!\langle Q(t)\rangle\!\rangle = \langle A_n(t_n)\ldots(\mathcal{R}A_i)(t)\ldots A_1(t_1)\rangle_B = \langle\!\langle(\mathcal{R}Q)(t)\rangle\!\rangle \tag{A2.5}$$

where $(\mathcal{R}Q)$ is defined in (4.3). In doing so we have reintroduced the summation over ρ and α and carried out some elementary transformations.

For $m \geq 2$ the expression $\exp(\psi)$ in (3.19) also depends on the maximum time argument t. Differentiating (4.1) with respect to t we get additional terms resulting from $\exp(\psi)$ which give rise to the double sum in (4.2). Thus for both $m = 1$ and $m \geq 2$ we arrive at equation (4.2) by straightforward differentiation of equation (4.1).

Appendix 3

Using the following abbreviations

$$F_{A_i}(t_i) = \sum_{\sigma_i=1}^{2} \varphi_i^{(\sigma_i)}A_i^{(\sigma_i)}(t_i); \qquad \varphi_i^{(1)} = -1, \qquad \varphi_i^{(2)} = \partial/\partial t_i \tag{A3.1}$$

$$A_i^{(1)}(t_i) = (\mathcal{R}A_i)(t_i), \qquad A_i^{(2)}(t_i) = A_i(t_i) \tag{A3.2}$$

together with (3.19) we arrive at

$$\begin{aligned} K(A_n, \ldots, A_1; t_n, \ldots, t_1) &= \sum_{\sigma_n=1}^{2} \cdots \sum_{\sigma_1=1}^{2} \varphi_n^{(\sigma_n)} \ldots \varphi_1^{(\sigma_1)} \langle A_n^{(\sigma_n)}(t_n) \ldots A_1^{(\sigma_1)}(t_1) \rangle_B \\ &= \sum_{\sigma_n=1}^{2} \cdots \sum_{\sigma_1=1}^{2} \varphi_n^{(\sigma_n)} \ldots \varphi_1^{(\sigma_1)} \exp(\psi) \exp(\chi) \widetilde{A_n^{(\sigma_n)}}(t_n) \ldots \widetilde{A_1^{(\sigma_1)}}(t_1). \end{aligned} \tag{A3.3}$$

For any sum in (A3.3) we get

$$\begin{aligned} &\sum_{\sigma_i=1}^{2} \varphi_i^{(\sigma_i)} \exp(\psi) \exp(\chi) \widetilde{A_n^{(\sigma_n)}}(t_n) \ldots \widetilde{A_i^{(\sigma_i)}}(t_i) \ldots \widetilde{A_1^{(\sigma_1)}}(t_1) \\ &\quad = -\exp(\psi) \exp(\chi) \widetilde{A_n^{(\sigma_n)}}(t_n) \ldots \widetilde{(\mathcal{R}A_i)}(t_i) \ldots \widetilde{A_1^{(\sigma_1)}}(t_1) \\ &\qquad + (\partial/\partial t_i) \exp(\psi) \exp(\chi) \widetilde{A_n^{(\sigma_n)}}(t_n) \ldots \tilde{A}_i(t_i) \ldots \widetilde{A_1^{(\sigma_1)}}(t_1) \\ &\quad = [(\partial/\partial t_i) \exp(\psi)] \exp(\chi) \widetilde{A_n^{(\sigma_n)}}(t_n) \ldots \tilde{A}_i(t_i) \ldots \widetilde{A_1^{(\sigma_1)}}(t_1) \end{aligned} \tag{A3.4}$$

and by continuing in this way we arrive at equation (5.3).

Appendix 4

In order to illustrate the generalised regression theorem we consider two simple examples. Firstly we investigate a harmonic oscillator (dynamic system) interacting with one bath described by (2.9*a*). We take

$$H_D = \omega_0 a^+ a, \qquad H_J = Ba^+ + B^+ a \tag{A4.1}$$

where a and a^+ fulfil the commutation relations (2.4) for Bose operators. From (4.3)-(4.5) we obtain

$$\mathcal{R}a = [-i\Omega - (\mu - \mu^T)]a \tag{A4.2}$$

$$\mathcal{R}_{21}\langle\langle a^+(t); a^+(t)\rangle\rangle = \mathcal{R}_{21}\langle\langle a(t); a(t)\rangle\rangle = 0 \tag{A4.3}$$

$$\mathcal{R}_{21}\langle\langle a^+(t); a(t)\rangle\rangle = 2\mu^T\langle\langle 1; 1\rangle\rangle, \qquad \mathcal{R}_{21}\langle\langle a(t); a^+(t)\rangle\rangle = 2\mu\langle\langle 1; 1\rangle\rangle \tag{A4.4}$$

where Ω is the renormalised oscillator frequency, and 1 is the unit operator. Inserting these results into (4.7) and (4.8) and solving the resulting differential equations we get

$$\langle\langle a(t)\rangle\rangle = \exp[(-i\Omega - \lambda)(t - t')]\langle\langle a(t')\rangle\rangle \tag{A4.5}$$

$$\langle\langle a^+(t); a^+(t)\rangle\rangle = \exp[(+2i\Omega - 2\lambda)(t - t')]\langle\langle a^+(t'); a^+(t')\rangle\rangle \tag{A4.6}$$

$$\langle\langle a^+(t); a(t)\rangle\rangle = \exp[-2\lambda(t - t')]\langle\langle a^+(t'); a(t')\rangle\rangle + n^T\{1 - \exp[-2\lambda(t - t')]\}\langle\langle 1; 1\rangle\rangle \tag{A4.7}$$

where t' is the second in height time argument in the CFD under consideration, $\lambda = \mu - \mu^T$ and $n^T = \mu^T/\lambda$. In the special case $\langle\langle a^+(t); a(t)\rangle\rangle \equiv \langle a^+(t)a^+(t')a(t')a(t)\rangle_B$

from (A4.7) we obtain

$$\langle a^+(t)a^+(t')a(t')a(t)\rangle_B$$
$$= \exp[-2\lambda(t-t')]\langle a^+(t')a^+(t')a(t')a(t')\rangle_B$$
$$+ n^T\{1-\exp[-2\lambda(t-t')]\}\langle a^+(t')a(t')\rangle_B. \tag{A4.8}$$

More generally the final result for $\langle a^+(t_4)a^+(t_3)a(t_2)a(t_1)\rangle_B$ where the t_i have no fixed relation with each other may be written as follows ($t_{min} = \min\{t_4, \ldots, t_1\}$, $\langle \ldots \rangle = Tr_D\rho_D\langle \ldots \rangle_B$):

$$\langle a^+(t_4)a^+(t_3)a(t_2)a(t_1)\rangle$$
$$= \exp[i\Omega(t_4+t_3-t_2-t_1)]$$
$$\times(\alpha(t_{min})\{\exp[-\lambda(|t_4-t_2|+|t_3-t_1|)]+\exp[-\lambda(|t_4-t_1|+|t_3-t_2|)]\}n^T$$
$$+\beta(t_{min})\exp[-\lambda(t_4+t_3+t_2+t_1-4t_{min})]) \tag{A4.9}$$

where $(\alpha(t) \equiv \langle a^+(t)a(t)\rangle$, $\beta(t) \equiv \langle a^+(t)a^+(t)a(t)a(t)\rangle - 2n^T\alpha(t))$

$$\alpha(t) = n^T + \exp(-2\lambda t)(\langle a^+a\rangle - n^T) \tag{A4.10}$$

$$\beta(t) = \exp(-4\lambda t)(\langle a^+a^+aa\rangle - 4n^T\langle a^+a\rangle + 2n^{T2})$$
$$+2\exp(-2\lambda t)n^T(\langle a^+a\rangle - n^T) \tag{A4.11}$$

and $\langle a^+a^+aa\rangle$, $\langle a^+a\rangle$ are the expectation values with respect to the total system at time zero.

As a second and more involved example we consider a two-level atom interacting with one bath described by (2.9*a*). We take

$$H_D = \omega_0(a_{22}-a_{11})/2, \qquad H_J = Ba_{21} + B^+a_{12} \tag{A4.12}$$

where $a_{ij} = |i\rangle\langle j|$ $(i,j = 1,2;\ a_{im}a_{jn} = \delta_{mj}a_{in})$ and $|1\rangle(|2\rangle)$ is the ground (excited) state of the two-level atom with the transition frequency ω_0, and we have $a_{11}+a_{22} = 1$. From (4.3)-(4.5) we obtain, for example

$$\mathcal{R}a_{12} = [-i\Omega - (\mu+\mu^T)]a_{12}; \qquad \mathcal{R}a_{22} = -2(\mu+\mu^T)a_{22} + 2\mu^T 1 \tag{A4.13}$$

$$\mathcal{R}_{21}\langle\langle a_{21}(t); a_{12}(t)\rangle\rangle = 2\mu^T\langle\langle n(t); n(t)\rangle\rangle \tag{A4.14}$$

where $n(t) = a_{22}(t) - a_{11}(t)$; and Ω is the renormalised transition frequency.

In order to determine for instance the CFD $\langle a_{21}(t)a_{22}(t')a_{12}(t)\rangle_B$ we first have to solve the following system of differential equations resulting from (4.8):

$$(d/dt)\langle\langle a_{21}(t); a_{12}(t)\rangle\rangle = -2(\mu+\mu^T)\langle\langle a_{21}(t); a_{12}(t)\rangle\rangle + 2\mu^T\langle\langle n(t); n(t)\rangle\rangle \tag{A4.15}$$

$$(d/dt)\langle\langle a_{12}(t); a_{21}(t)\rangle\rangle = -2(\mu+\mu^T)\langle\langle a_{12}(t); a_{21}(t)\rangle\rangle + 2\mu\langle\langle n(t); n(t)\rangle\rangle \tag{A4.16}$$

$$(d/dt)\langle\langle n(t); n(t)\rangle\rangle = -4(\mu+\mu^T)\langle\langle n(t); n(t)\rangle\rangle + 8\mu\langle\langle a_{21}(t); a_{12}(t)\rangle\rangle$$
$$+8\mu^T\langle\langle a_{12}(t); a_{21}(t)\rangle\rangle + S(t) \tag{A4.17}$$

with the initial conditions

$$\langle\langle a_{21}(t'); a_{12}(t')\rangle\rangle \equiv \langle a_{21}(t')a_{22}(t')a_{12}(t')\rangle_B = 0 \tag{A4.18}$$

$$\langle\langle a_{12}(t'); a_{21}(t')\rangle\rangle \equiv \langle a_{12}(t')a_{22}(t')a_{21}(t')\rangle_B = \langle a_{11}(t')\rangle_B \tag{A4.19}$$

$$\langle\langle n(t'); n(t')\rangle\rangle \equiv \langle n(t')a_{22}(t')n(t')\rangle_B = \langle a_{22}(t')\rangle_B \tag{A4.20}$$

where

$$S(t)=2(\mu^T-\mu)(\langle\!\langle 1;n(t)\rangle\!\rangle+\langle\!\langle n(t);1\rangle\!\rangle);\qquad S(t')=4(\mu^T-\mu)\langle a_{22}(t')\rangle_B. \tag{A4.21}$$

On the other hand, in order to determine $\langle a_{21}(t')a_{22}(t)a_{12}(t')\rangle_B$, we only have to solve

$$(\mathrm{d}/\mathrm{d}t)\langle\!\langle a_{22}(t)\rangle\!\rangle=-2(\mu+\mu^T)\langle\!\langle a_{22}(t)\rangle\!\rangle+2\mu^T\langle\!\langle 1\rangle\!\rangle \tag{A4.22}$$

with the initial condition

$$\langle\!\langle a_{22}(t')\rangle\!\rangle\equiv\langle a_{21}(t')a_{22}(t')a_{12}(t')\rangle_B=0 \tag{A4.23}$$

where $\langle\!\langle 1\rangle\!\rangle\equiv\langle a_{21}(t')1a_{12}(t')\rangle_B=\langle a_{22}(t')\rangle_B$.

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